Fifth Edition

Engineering Mathematics

A Foundation for Electronic, Electrical, Communications and Systems Engineers

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9 Complex numbers

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9.1 INTRODUCTION

Complex numbers often seem strange when first encountered but it is worth persevering with them because they provide a powerful mathematical tool for solving several engineering problems. One of the main applications is to the analysis of alternating current (a.c.) circuits. Engineers are very interested in these because the mains supply is itself a.c., and electricity generation and transportation are dominated by a.c. voltages and currents.

A great deal of signal analysis and processing uses mathematical models based on complex numbers because they allow the manipulation of sinusoidal quantities to be undertaken more easily. Furthermore, the design of filters to be used in communications equipment relies heavily on their use.

One area of particular relevance is control engineering – so much so that control engineers often prefer to think of a control system in terms of a 'complex plane' representation rather than a 'time domain' representation. We will develop these concepts in this and subsequent chapters.

9.2 COMPLEX NUMBERS

We have already examined quadratic equations such as

$$x^2 - x - 6 = 0 \tag{9.1}$$

and have met techniques for finding the roots of such equations. The formula for obtaining the roots of a quadratic equation $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{9.2}$$

Applying this formula to Equation (9.1), we find

$$\begin{aligned} x &= \frac{+1 \pm \sqrt{(-1)^2 - 4(1)(-6)}}{2} \\ &= \frac{1 \pm \sqrt{25}}{2} \\ &= \frac{1 \pm 5}{2} \end{aligned}$$

so that $x = 3$ and $x = -2$ are the two roots. However, if we try to apply the formula to the equation

$$2x^2 + 2x + 5 = 0$$

we find

$$x = \frac{-2 \pm \sqrt{-36}}{4}$$

A problem now arises in that we need to find the square root of a negative number. We know from experience that squaring both positive and negative numbers yields a positive result; thus,

$$6^2 = 36 \qquad \text{and} \qquad (-6)^2 = 36$$

so that there is no real number whose square is -36. In the general case, if $ax^2 + bx + c = 0$, we see by examining the square root in Equation (9.2) that this problem will always arise whenever $b^2 - 4ac < 0$. Nevertheless, it turns out to be very useful to invent a technique for dealing with such situations, leading to the theory of complex numbers.

To make progress we introduce a number, denoted j, with the property that

$$\boxed{\mathrm{j}^2 = -1}$$

We have already seen that using the real number system we cannot obtain a negative number by squaring a real number so the number j is not real – we say it is **imaginary**. This imaginary number has a very useful role to play in engineering mathematics. Using it we can now formally write down an expression for the square root of any negative number. Thus,

$$\begin{aligned} \sqrt{-36} &= \sqrt{36 \times (-1)} \\ &= \sqrt{36 \times \mathrm{j}^2} \\ &= 6\mathrm{j} \end{aligned}$$

Returning to the solution of the quadratic equation $2x^2 + 2x + 5 = 0$, we find

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{-36}}{4} \\ &= \frac{-2 \pm 6\mathrm{j}}{4} \\ &= \frac{-1 \pm 3\mathrm{j}}{2} \end{aligned}$$

We have found two roots, namely $x = -\frac{1}{2} + \frac{3}{2}\mathrm{j}$ and $x = -\frac{1}{2} - \frac{3}{2}\mathrm{j}$. These numbers are called **complex numbers** and we see that they are made up of two parts – a **real part** and an **imaginary part**. For the first complex number the real part is $-\frac{1}{2}$ and the imaginary part is $\frac{3}{2}$. For the second complex number the real part is $-\frac{1}{2}$ and the imaginary part is $-\frac{3}{2}$. In a more general case we usually use the letter z to denote a complex number with real part a and imaginary part b, so $z = a + b\mathrm{j}$. We write $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$, and denote the set of all complex numbers by $\mathbb{C}$. Note that $a, b \in \mathbb{R}$ whereas $z \in \mathbb{C}$.

> $z = a + b\mathrm{j}$, z is a member of the set of complex numbers, that is $z \in \mathbb{C}$
> $a = \mathrm{Re}(z) \qquad b = \mathrm{Im}(z)$

Complex numbers which have a zero imaginary part are purely real and hence all real numbers are also complex numbers, that is $\mathbb{R} \subset \mathbb{C}$.

Example 9.1 Solve the quadratic equation $2s^2 - 3s + 7 = 0$.

Solution Using the formula for solving a quadratic equation we find

$$\begin{aligned} s &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(7)}}{2(2)} \\ &= \frac{3 \pm \sqrt{-47}}{4} \\ &= \frac{3 \pm \sqrt{47}\mathrm{j}}{4} \\ &= 0.75 \pm 1.71\mathrm{j} \end{aligned}$$

Using the fact that $\mathrm{j}^2 = -1$ we can develop other quantities.

Example 9.2 Simplify the expression j^3.

Solution We have

$$\begin{aligned} \mathrm{j}^3 &= \mathrm{j}^2 \times \mathrm{j} \\ &= (-1) \times \mathrm{j} \\ &= -\mathrm{j} \end{aligned}$$

9.2.1 The complex conjugate

If $z = a + b\mathrm{j}$, we define its **complex conjugate** to be the number $\bar{z} = a - b\mathrm{j}$; that is, we change the sign of the imaginary part.

Example 9.3 Write down the complex conjugates of

(a) $-7 + \mathrm{j}$ (b) $6 - 5\mathrm{j}$ (c) 6 (d) j

Solution To find the complex conjugates of the given numbers we change the sign of the imaginary parts. A purely real number has an imaginary part 0. We find

(a) $-7 - \mathrm{j}$ (b) $6 + 5\mathrm{j}$ (c) 6, there is no imaginary part to alter (d) $-\mathrm{j}$

We recall that the solution of the quadratic equation $2x^2 + 2x + 5 = 0$ yielded the two complex numbers $-\frac{1}{2} + \frac{3}{2}\mathrm{j}$ and $-\frac{1}{2} - \frac{3}{2}\mathrm{j}$, and note that these form a complex conjugate pair. This illustrates a more general result:

> When the polynomial equation $P(x) = 0$ has real coefficients, any complex roots will always occur in **complex conjugate pairs**.

Consider the following example.

Example 9.4 Show that the equation $x^3 - 7x^2 + 19x - 13 = 0$ has a root at $x = 1$ and find the other roots.

Solution If we let $P(x) = x^3 - 7x^2 + 19x - 13$, then $P(1) = 1 - 7 + 19 - 13 = 0$ so that $x = 1$ is a root. This means that $x - 1$ must be a factor of $P(x)$ and so we can express $P(x)$ in the form

$$\begin{aligned} P(x) = x^3 - 7x^2 + 19x - 13 &= (x-1)(\alpha x^2 + \beta x + \gamma) \\ &= \alpha x^3 + (\beta - \alpha)x^2 + (\gamma - \beta)x - \gamma \end{aligned}$$

where α, β and γ are coefficients to be determined. Comparing the coefficients of x^3 we find $\alpha = 1$. Comparing the constant coefficients we find $\gamma = 13$. Finally, comparing coefficients of x we find $\beta = -6$, and hence

$$P(x) = x^3 - 7x^2 + 19x - 13 = (x-1)(x^2 - 6x + 13)$$

The other two roots of $P(x) = 0$ are found by solving the quadratic equation $x^2 - 6x + 13 = 0$, that is

$$x = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2\mathrm{j}$$

and again we note that the complex roots occur as a complex conjugate pair. This illustrates the general result given in Section 1.4 that an nth-degree polynomial has n roots.

EXERCISES 9.2

1 Solve the following equations:

(a) $x^2 + 1 = 0$ (b) $x^2 + 4 = 0$
(c) $3x^2 + 7 = 0$ (d) $x^2 + x + 1 = 0$
(e) $\dfrac{x^2}{2} - x + 2 = 0$
(f) $-x^2 - 3x - 4 = 0$
(g) $2x^2 + 3x + 3 = 0$
(h) $x^2 + 3x + 4 = 0$

2 Solve the cubic equation

$$3x^3 - 11x^2 + 16x - 12 = 0$$

given that one of the roots is $x = 2$.

3 Write down the complex conjugates of the following complex numbers.

(a) $-11 - 8\text{j}$ (b) $5 + 3\text{j}$ (c) $\dfrac{1}{2}\text{j}$ (d) -17
(e) $\cos \omega t + \text{j}\sin \omega t$ (f) $\cos \omega t - \text{j}\sin \omega t$
(g) $-0.333\text{j} + 1$

4 Recall from Chapter 2 that the poles of a rational function $R(x) = P(x)/Q(x)$ are those values of x for which $Q(x) = 0$. Find any poles of

(a) $\dfrac{x}{x-3}$ (b) $\dfrac{3x}{x^2+1}$ (c) $\dfrac{3}{x^2+x+1}$

5 Solve the equation $s^2 + 2s + 5 = 0$.

6 Express as a complex number

(a) j^4 (b) j^5 (c) j^6

7 State $\text{Re}(z)$ and $\text{Im}(z)$ where

(a) $z = 7 + 11\text{j}$ (b) $z = -6 + \text{j}$
(c) $z = 0$ (d) $z = \dfrac{1+\text{j}}{2}$
(f) $z = \text{j}$ (e) $z = \text{j}^2$

Solutions

1 (a) $\pm\text{j}$ (b) $\pm 2\text{j}$
(c) $\pm\sqrt{7/3}\,\text{j}$ (d) $-1/2 \pm (\sqrt{3}/2)\text{j}$
(e) $1 \pm \sqrt{3}\,\text{j}$ (f) $-3/2 \pm (\sqrt{7}/2)\text{j}$
(g) $-3/4 \pm (\sqrt{15}/4)\text{j}$ (h) $-3/2 \pm (\sqrt{7}/2)\text{j}$

2 $2,\ 5/6 \pm (\sqrt{47}/6)\text{j}$

3 (a) $-11 + 8\text{j}$ (b) $5 - 3\text{j}$ (c) $-\dfrac{1}{2}\text{j}$
(d) -17 (e) $\cos \omega t - \text{j}\sin \omega t$
(f) $\cos \omega t + \text{j}\sin \omega t$ (g) $0.333\text{j} + 1$

4 (a) $x = 3$ (b) $x = \pm\text{j}$
(c) $x = -1/2 \pm (\sqrt{3}/2)\text{j}$

5 $s = -1 \pm 2\text{j}$

6 (a) 1 (b) j (c) -1

7 (a) 7, 11 (b) -6, 1 (c) 0, 0
(d) $\dfrac{1}{2}, \dfrac{1}{2}$ (e) 0, 1 (f) -1, 0

9.3 OPERATIONS WITH COMPLEX NUMBERS

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

Example 9.5 Find x and y so that $x + 6\text{j}$ and $3 - y\text{j}$ represent the same complex number.

Solution If both quantities represent the same complex number we have

$$x + 6\text{j} = 3 - y\text{j}$$

Since the real parts must be equal we can equate them, that is

$$x = 3$$

Similarly, we find, by equating imaginary parts

$$6 = -y$$

so that $y = -6$.

The operations of addition, subtraction, multiplication and division can all be performed on complex numbers.

9.3.1 Addition and subtraction

To add two complex numbers we simply add the real parts and add the imaginary parts; to subtract a complex number from another we subtract the corresponding real parts and subtract the corresponding imaginary parts as shown in Example 9.6.

Example 9.6 If $z_1 = 3 - 4\mathrm{j}$ and $z_2 = 4 + 2\mathrm{j}$ find $z_1 + z_2$ and $z_1 - z_2$.

Solution

$$\begin{aligned} z_1 + z_2 &= (3 - 4\mathrm{j}) + (4 + 2\mathrm{j}) \\ &= (3 + 4) + (-4 + 2)\mathrm{j} \\ &= 7 - 2\mathrm{j} \\ z_1 - z_2 &= (3 - 4\mathrm{j}) - (4 + 2\mathrm{j}) \\ &= (3 - 4) + (-4 - 2)\mathrm{j} \\ &= -1 - 6\mathrm{j} \end{aligned}$$

9.3.2 Multiplication

We can multiply a complex number by a real number. Both the real and imaginary parts of the complex number are multiplied by the real number. Thus $3(4 - 6\mathrm{j}) = 12 - 18\mathrm{j}$.

To multiply two complex numbers we use the fact that $\mathrm{j}^2 = -1$.

Example 9.7 If $z_1 = 2 - 2\mathrm{j}$ and $z_2 = 3 + 4\mathrm{j}$, find $z_1 z_2$.

Solution

$$z_1 z_2 = (2 - 2\mathrm{j})(3 + 4\mathrm{j})$$

Removing brackets we find

$$\begin{aligned} z_1 z_2 &= 6 - 6\mathrm{j} + 8\mathrm{j} - 8\mathrm{j}^2 \\ &= 6 - 6\mathrm{j} + 8\mathrm{j} + 8 \qquad \text{using } \mathrm{j}^2 = -1 \\ &= 14 + 2\mathrm{j} \end{aligned}$$

Example 9.8 If $z = 3 - 2\mathrm{j}$ find $z\bar{z}$.

Solution If $z = 3 - 2\mathrm{j}$ then its conjugate is $\bar{z} = 3 + 2\mathrm{j}$. Therefore,

$$\begin{aligned} z\bar{z} &= (3 - 2\mathrm{j})(3 + 2\mathrm{j}) \\ &= 9 - 6\mathrm{j} + 6\mathrm{j} - 4\mathrm{j}^2 \\ &= 9 - 4\mathrm{j}^2 \\ &= 13 \end{aligned}$$

We see that the answer is a real number.

Whenever we multiply a complex number by its conjugate the answer is a real number. Thus if $z = a + b\mathrm{j}$

$$\begin{aligned} z\bar{z} &= (a + b\mathrm{j})(a - b\mathrm{j}) \\ &= a^2 + ba\mathrm{j} - ab\mathrm{j} - b^2\mathrm{j}^2 \\ &= a^2 + b^2 \end{aligned}$$

If $z = a + b\mathrm{j}$ then $z\bar{z} = a^2 + b^2$

9.3.3 Division

To divide two complex numbers it is necessary to make use of the complex conjugate. We multiply both the numerator and denominator by the conjugate of the denominator and then simplify the result.

Example 9.9 If $z_1 = 2 + 9\mathrm{j}$ and $z_2 = 5 - 2\mathrm{j}$ find $\dfrac{z_1}{z_2}$.

Solution We seek $\dfrac{2 + 9\mathrm{j}}{5 - 2\mathrm{j}}$. The complex conjugate of the denominator is $5 + 2\mathrm{j}$, so we multiply both numerator and denominator by this quantity. The effect of this is to leave the value of $\dfrac{z_1}{z_2}$ unaltered since we have only multiplied by 1. Therefore,

$$\begin{aligned} \frac{z_1}{z_2} = \frac{2 + 9\mathrm{j}}{5 - 2\mathrm{j}} &= \frac{(2 + 9\mathrm{j})}{(5 - 2\mathrm{j})}\frac{(5 + 2\mathrm{j})}{(5 + 2\mathrm{j})} \\ &= \frac{10 + 45\mathrm{j} + 4\mathrm{j} + 18\mathrm{j}^2}{25 + 4} = \frac{-8 + 49\mathrm{j}}{29} \\ &= -\frac{8}{29} + \frac{49}{29}\mathrm{j} \end{aligned}$$

The multiplication of two conjugates in the denominator allows a useful simplification. We see that the effect of multiplying by the conjugate of the denominator is to make the denominator of the solution purely real.

If $z_1 = x_1 + \mathrm{j}y_1$ and $z_2 = x_2 + \mathrm{j}y_2$ then the quotient $\dfrac{z_1}{z_2}$ is found by multiplying both numerator and denominator by the conjugate of the denominator, that is

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{x_1 + \mathrm{j}y_1}{x_2 + \mathrm{j}y_2} \\ &= \frac{x_1 + \mathrm{j}y_1}{x_2 + \mathrm{j}y_2} \times \frac{x_2 - \mathrm{j}y_2}{x_2 - \mathrm{j}y_2} \\ &= \frac{x_1x_2 + y_1y_2 + \mathrm{j}(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} \end{aligned}$$

EXERCISES 9.3

1 If $z_1 = 3 + 2\text{j}$ and $z_2 = 4 - 8\text{j}$ find

(a) $z_1 + z_2$ (b) $z_1 - z_2$ (c) $z_2 - z_1$

2 Express the following in the form $a + b\text{j}$:

(a) $\dfrac{1}{1+\text{j}}$ (b) $\dfrac{-2}{\text{j}}$

(c) $\dfrac{1}{\text{j}} + \dfrac{1}{2-\text{j}}$ (d) $\dfrac{\text{j}}{1+\text{j}}$

(e) $\dfrac{3}{3+2\text{j}} + \dfrac{1}{5-\text{j}}$

3 Express the following in the form $a + b\text{j}$:

(a) $\dfrac{2}{1-\text{j}}$ (b) $\dfrac{-2+3\text{j}}{\text{j}}$

(c) $3\text{j}(4 - 2\text{j})$ (b) $(7 - 2\text{j})(5 + 6\text{j})$

(e) $\dfrac{5+3\text{j}}{2+2\text{j}}$

4 Find a quadratic equation whose roots are $1 - 3\text{j}$ and $1 + 3\text{j}$.

5 If $(x + \text{j}y)^2 = 3 + 4\text{j}$, find x and y, where $x, y \in \mathbb{R}$.

6 Find the real and imaginary parts of

(a) $\dfrac{2}{4+\text{j}} - \dfrac{3}{2-\text{j}}$ (b) $\text{j}^4 - \text{j}^5$

(c) $\dfrac{1}{\text{j}} + \text{j}$ (d) $\dfrac{1}{\text{j}^3 - 3\text{j}}$

Solutions

1 (a) $7 - 6\text{j}$ (b) $-1 + 10\text{j}$ (c) $1 - 10\text{j}$

2 (a) $\dfrac{1}{2} - \dfrac{1}{2}\text{j}$ (b) 2j (c) $\dfrac{2}{5} - \dfrac{4}{5}\text{j}$

(d) $\dfrac{1}{2} + \dfrac{1}{2}\text{j}$ (e) $\dfrac{23 - 11\text{j}}{26}$

3 (a) $1 + \text{j}$ (b) $3 + 2\text{j}$ (c) $6 + 12\text{j}$

(d) $47 + 32\text{j}$ (e) $2 - \dfrac{1}{2}\text{j}$

4 $x^2 - 2x + 10 = 0$

5 $x = 2, y = 1; x = -2, y = -1$

6 (a) $-\dfrac{62}{85}, -\dfrac{61}{85}$

(b) $1, -1$

(c) $0, 0$

(d) $0, \dfrac{1}{4}$

Technical Computing Exercises 9.3

1 Many technical computing languages allow the user to input and manipulate complex numbers. Investigate how the complex number $a + b\text{j}$ is input to the software to which you have access.

2 Use technical computing software to simplify the following complex numbers:

(a) $(1 + \text{j})^5$ (b) $\dfrac{3\text{j}}{(1 - 2\text{j})^7}$

(c) $\dfrac{4}{(3+\text{j})^3} - \dfrac{7}{(2-3\text{j})^2}$

3 Solve the polynomial equations

(a) $x^3 + 7x^2 + 9x + 63 = 0$

(b) $x^4 + 15x^2 - 16 = 0$

You may recall from Chapter 1 that in MATLAB® the `roots` function can be used to solve polynomial equations with real roots. This function can also calculate complex roots.

4 Use a package to find the real and imaginary parts of $\dfrac{15+\text{j}}{7-5\text{j}}$.

5 A common requirement in control theory is to find the poles of a rational function. Use a package to find the poles of the function

$$G(s) = \frac{0.5}{s^3 + 2s^2 + 0.2s + 0.0556}$$

To solve this problem automatically may require the use of specialist toolboxes in MATLAB® or GNU Octave which are not part of the core program. For example, the Signal Processing toolbox in MATLAB® has the function `tf2zp` which converts a transfer function expressed as a ratio of polynomials to pole/zero form.

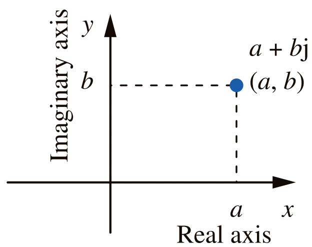

Figure 9.1
Argand diagram.

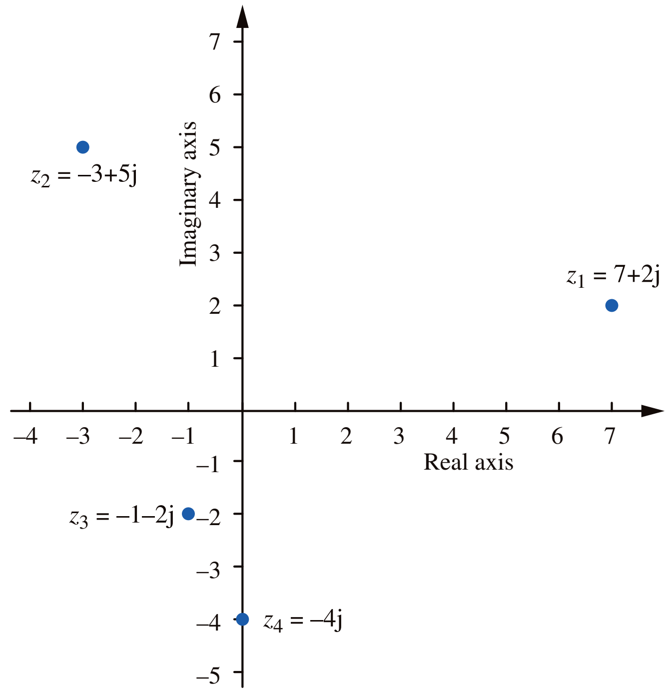

Figure 9.2
Argand diagram for Example 9.10.

9.4 GRAPHICAL REPRESENTATION OF COMPLEX NUMBERS

Given a complex number $z = a + b\mathrm{j}$ we can obtain a useful graphical interpretation of it by plotting the real part on the horizontal axis and the imaginary part on the vertical axis and obtain a unique point in the x–y plane (Figure 9.1). We call the x axis the **real axis** and the y axis the **imaginary axis**, and the whole picture an **Argand diagram**. In this context, the x–y plane is often referred to as the **complex plane**.

Example 9.10 Plot the complex numbers $z_1 = 7 + 2\mathrm{j}$, $z_2 = -3 + 5\mathrm{j}$, $z_3 = -1 - 2\mathrm{j}$ and $z_4 = -4\mathrm{j}$ on an Argand diagram.

Solution The Argand diagram is shown in Figure 9.2.

EXERCISES 9.4

1 Plot the following complex numbers on an Argand diagram:

(a) $z_1 = -3 - 3\mathrm{j}$
(b) $z_2 = 7 + 2\mathrm{j}$
(c) $z_3 = 3$
(d) $z_4 = -0.5\mathrm{j}$
(e) $z_5 = -2$

2 (a) Plot the complex number $z = 1 + \mathrm{j}$ on an Argand diagram.

(b) Simplify the complex number $\mathrm{j}(1 + \mathrm{j})$ and plot the result on your Argand diagram. Observe that the effect of multiplying the complex number by j is to rotate the complex number through an angle of $\pi/2$ radians anticlockwise about the origin.

Solutions

1 See Figure S.17.

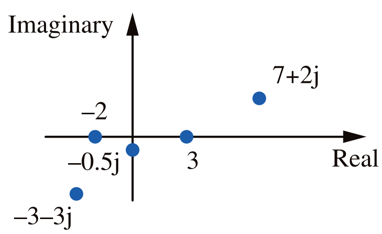

Figure S.17

2 (a) See Figure S.18. (b) $j(1 + j) = -1 + j$

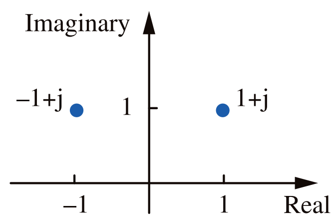

Figure S.18

9.5 POLAR FORM OF A COMPLEX NUMBER

It is often useful to exchange **Cartesian coordinates** (a, b) for **polar coordinates** r and θ as depicted in Figure 9.3.

From Figure 9.3 we note that

$$\cos\theta = \frac{a}{r} \qquad \sin\theta = \frac{b}{r}$$

and so,

$$a = r\cos\theta \qquad b = r\sin\theta$$

Furthermore,

$$\tan\theta = \frac{b}{a}$$

Using Pythagoras's theorem we obtain $r = \sqrt{a^2 + b^2}$. By finding r and θ we can express the complex number $z = a + b\mathrm{j}$ in **polar form** as

$$z = r\cos\theta + \mathrm{j}r\sin\theta = r(\cos\theta + \mathrm{j}\sin\theta)$$

which we often abbreviate to $z = r\angle\theta$. Clearly, r is the 'distance' of the point (a, b) from the origin and is called the **modulus** of the complex number z. The modulus is always a non-negative number and is denoted $|z|$. The angle is conventionally measured from the positive x axis. Angles measured in an anticlockwise sense are regarded as positive while those measured in a clockwise sense are regarded as negative. The angle θ is called the **argument** of z, denoted $\arg(z)$. Since adding or subtracting multiples of 2π from θ will result in the 'arm' in Figure 9.3 being in the same position, the argument can have many values. Usually we shall choose θ to satisfy $-\pi < \theta \leqslant \pi$.

Cartesian form: $z = a + b\mathrm{j}$

Polar form: $z = r(\cos\theta + \mathrm{j}\sin\theta) = r\angle\theta$

$$|z| = r = \sqrt{a^2 + b^2}$$

$$a = r\cos\theta \qquad b = r\sin\theta \qquad \tan\theta = \frac{b}{a}$$

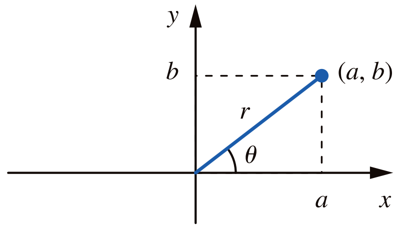

Figure 9.3
Polar and Cartesian forms of a complex number.

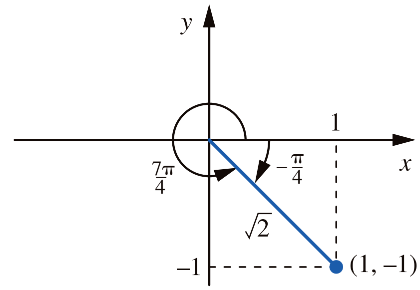

Figure 9.4
Argand diagram depicting $z = 1 - \mathrm{j}$ in Example 9.11.

Note that

$$\begin{aligned} r\angle(-\theta) &= r(\cos(-\theta) + \mathrm{j}\sin(-\theta)) \\ &= r(\cos\theta - \mathrm{j}\sin\theta) \\ &= \bar{z} \end{aligned}$$

If $z = a + b\mathrm{j}$ then $\bar{z} = a - b\mathrm{j}$ and $\bar{z} = r\angle(-\theta)$.

Example 9.11 Depict the complex number $z = 1 - \mathrm{j}$ on an Argand diagram and convert it into polar form.

Solution The real part of z is 1 and the imaginary part is -1. We therefore plot a point in the x–y plane with $x = 1$ and $y = -1$ as shown in Figure 9.4.

From Figure 9.4 we see that $r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$ and $\theta = -45°$ or $-\pi/4$ radians. Therefore $z = 1 - \mathrm{j} = \sqrt{2}\angle(-\pi/4)$.

To express a complex number in polar form it is essential to draw an Argand diagram and not simply quote formulae, as the following example will show.

Example 9.12 Express $z = -1 - \mathrm{j}$ in polar form.

Solution If we use the formula $|z| = r = \sqrt{a^2 + b^2}$, we find that $r = \sqrt{2}$. Using $\tan\theta = b/a$, we find that $\tan\theta = -1/-1 = 1$ so that you may be tempted to take $\theta = \pi/4$. Figure 9.5 shows the Argand diagram and it is clear that $\theta = -3\pi/4$. Therefore, $z = -1 - \mathrm{j} = \sqrt{2}\angle -3\pi/4$, and we see the importance of drawing an Argand diagram.

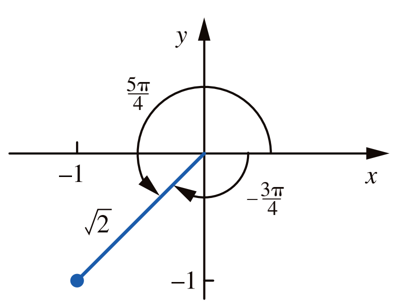

Figure 9.5
Argand diagram depicting $z = -1 - \mathrm{j}$ in Example 9.12.

9.5.1 Multiplication and division in polar form

The polar form may seem more complicated than the Cartesian form but it is often more useful. For example, suppose we want to multiply the complex numbers

$$z_1 = r_1(\cos\theta_1 + \mathrm{j}\sin\theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos\theta_2 + \mathrm{j}\sin\theta_2)$$

We find

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + \mathrm{j}\sin\theta_1)r_2(\cos\theta_2 + \mathrm{j}\sin\theta_2) \\ &= r_1 r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + \mathrm{j}(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)\} \end{aligned}$$

which can be written as

$$r_1 r_2\{\cos(\theta_1 + \theta_2) + \mathrm{j}\sin(\theta_1 + \theta_2)\}$$

using the trigonometric identities of Section 3.6. This is a new complex number which, if we compare with the general form $r(\cos\theta + \mathrm{j}\sin\theta)$, we see has a modulus of $r_1 r_2$ and an argument of $\theta_1 + \theta_2$. To summarize: to multiply two complex numbers we multiply their moduli and add their arguments, that is

$$\boxed{z_1 z_2 = r_1 r_2 \angle(\theta_1 + \theta_2)}$$

Example 9.13 If $z_1 = 3\angle\pi/3$ and $z_2 = 4\angle\pi/6$ find $z_1 z_2$.

Solution Multiplying the moduli we find $r_1 r_2 = 12$, and adding the arguments we find $\theta_1 + \theta_2 = \pi/2$. Therefore $z_1 z_2 = 12\angle\pi/2$.

A similar development shows that to divide two complex numbers we divide their moduli and subtract their arguments, that is

$$\boxed{\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2)}$$

Example 9.14 If $z_1 = 3\angle\pi/3$ and $z_2 = 4\angle\pi/6$ find z_1/z_2.

Solution Dividing the respective moduli, we find $r_1/r_2 = 3/4$ and subtracting the arguments, $\pi/3 - \pi/6 = \pi/6$. Hence $z_1/z_2 = 0.75\angle\pi/6$.

EXERCISES 9.5

1. Mark on an Argand diagram points representing $z_1 = 3 - 2\mathrm{j}$, $z_2 = -\mathrm{j}$, $z_3 = \mathrm{j}^2$, $z_4 = -2 - 4\mathrm{j}$ and $z_5 = 3$. Find the modulus and argument of each complex number.

2. Express the following complex numbers in polar form:

   (a) $3 - \mathrm{j}$ (b) 2 (c) $-\mathrm{j}$ (d) $-5 + 12\mathrm{j}$

3. Find the modulus and argument of (a) $z_1 = -\sqrt{3} + \mathrm{j}$ and (b) $z_2 = 4 + 4\mathrm{j}$. Hence express $z_1 z_2$ and z_1/z_2 in polar form.

4. Express $\sqrt{2}\angle\pi/4$, $2\angle\pi/6$ and $2\angle -\pi/6$ in Cartesian form $a + b\mathrm{j}$.

5. Prove the result $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\angle\theta_1 - \theta_2$.

6 Express $z = \dfrac{1}{j\omega C}$, where ω and C are real constants, in the form $a + bj$. Plot z on an Argand diagram.

7 If $z_1 = 4(\cos 40^\circ + j\sin 40^\circ)$ and $z_2 = 3(\cos 70^\circ + j\sin 70^\circ)$, express z_1z_2 and z_1/z_2 in polar form.

8 Simplify

$$\frac{(\sqrt{2}\angle(5\pi/4))^2(2\angle(-\pi/3))^2}{2\angle(-\pi/6)}$$

Solutions

1 $|z_1| = \sqrt{13}$, $\arg(z_1) = -0.5880$
$|z_2| = 1$, $\arg(z_2) = -\pi/2$
$|z_3| = 1$, $\arg(z_3) = \pi$
$|z_4| = \sqrt{20}$, $\arg(z_4) = -2.0344$
$|z_5| = 3$, $\arg(z_5) = 0$

2 (a) $\sqrt{10}\angle -0.3218$ (b) $2\angle 0$
(c) $1\angle -\pi/2$ (d) $13\angle 1.9656$

3 (a) $2, 5\pi/6$
(b) $4\sqrt{2}, \pi/4$

$$z_1z_2 = 8\sqrt{2}\angle 13\pi/12$$
$$z_1/z_2 = \frac{\sqrt{2}}{4}\angle 7\pi/12$$

4 $1 + j, \sqrt{3} + j, \sqrt{3} - j$

6 $-\dfrac{j}{\omega C}$

7 $z_1z_2 = 12(\cos 110^\circ + j\sin 110^\circ)$
$z_1/z_2 = \dfrac{4}{3}(\cos 30^\circ - j\sin 30^\circ)$

8 4

9.6 VECTORS AND COMPLEX NUMBERS

It is often convenient to represent complex numbers by vectors in the x–y plane. Figure 9.6(a) shows the complex number $z = a + jb$. Figure 9.6(b) shows the equivalent vector. Figure 9.7 shows the complex numbers $z_1 = 2 + j$ and $z_2 = 1 + 3j$.

If we now evaluate $z_3 = z_1 + z_2$ we find $z_3 = 3 + 4j$ which is also shown. If we form a parallelogram, two sides of which are the representations of z_1 and z_2, we find that z_3 is the diagonal of the parallelogram. If we regard z_1 and z_2 as vectors in the plane we see that there is a direct analogy between the triangle law of vector addition (see Section 7.2.3) and the addition of complex numbers.

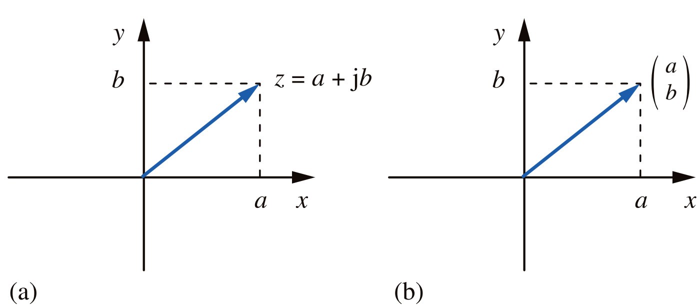

Figure 9.6
The complex number $z = a + jb$ and its equivalent vector.

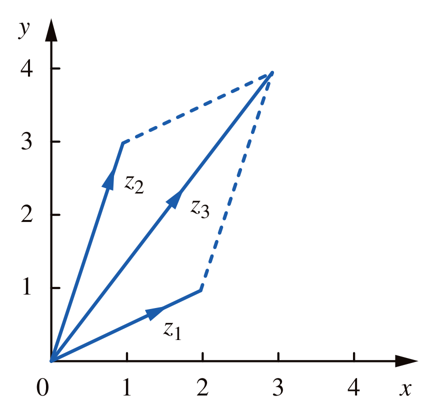

Figure 9.7
Vector addition in the complex plane.

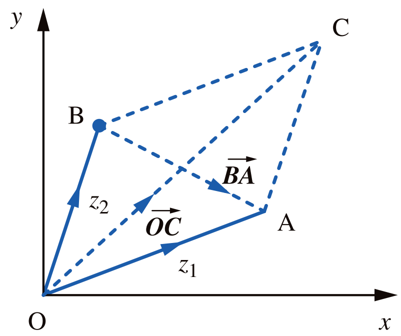

Figure 9.8
Vector addition and subtraction.

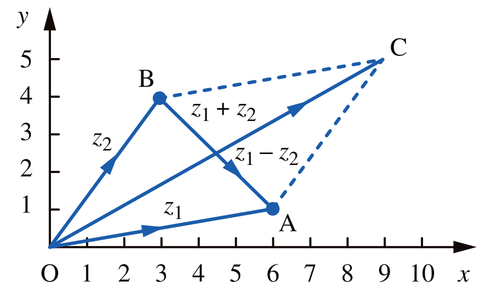

Figure 9.9
Diagram for Example 9.15.

More generally, if z_1 and z_2 are any complex numbers represented on an Argand diagram by the vectors $\vec{OA}$ and $\vec{OB}$ (Figure 9.8) then upon completing the parallelogram OBCA, the sum $z_1 + z_2$ is represented by the vector $\vec{OC}$. We can also obtain a representation of the difference of two complex numbers in the following way. If we write

$$\begin{aligned} z_3 &= z_1 - z_2 \\ &= z_1 + (-z_2) \end{aligned}$$

and note that if z_2 is represented by the vector $\vec{OB}$, then $-z_2$ is represented by the vector $-\vec{OB} = \vec{BO} = \vec{CA}$. The complex number z_1 is represented by $\vec{OA} = \vec{BC}$, so that $z_1 + (-z_2) = \vec{BC} + \vec{CA} = \vec{BA}$. Thus the difference $z_1 - z_2$ is represented by the diagonal BA. To summarize, the sum and difference of z_1 and z_2 are represented by the two diagonals of the parallelogram OBCA.

Example 9.15 Represent $z_1 = 6 + \mathrm{j}$ and $z_2 = 3 + 4\mathrm{j}$, and their sum and difference, on an Argand diagram.

Solution We draw vectors $\vec{OA}$ and $\vec{OB}$ representing z_1 and z_2, respectively (Figure 9.9). Then we complete the parallelogram OACB as shown. The sum $z_1 + z_2$ is then represented by $\vec{OC}$ and the difference $z_1 - z_2$ by $\vec{BA}$. It is easy to see that the vector $\vec{BA} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$, that is it represents the complex number $3 - 3\mathrm{j}$, while $\vec{OC} = \begin{pmatrix} 9 \\ 5 \end{pmatrix}$, that is it represents $9 + 5\mathrm{j}$, so that $z_1 + z_2 = 9 + 5\mathrm{j}$ and $z_1 - z_2 = 3 - 3\mathrm{j}$.

9.7 THE EXPONENTIAL FORM OF A COMPLEX NUMBER

You will recall from Chapter 6 that many functions possess a power series expansion, that is the function can be expressed as the sum of a sequence of terms involving integer powers of x. For example,

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and this representation is valid for any real value of x. The expression on the r.h.s. is, of course, an infinite sum but its terms get smaller and smaller, and as more are included, the sum we obtain approaches e^x. Other examples of power series include

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{9.3}$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \tag{9.4}$$

which are also valid for any real value of x. It is useful to extend the range of applicability of these power series by allowing x to be a complex number. That is, we define the function e^z to be

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

and theory beyond the scope of this book can be used to show that this representation is valid for all complex numbers z.

We have already seen that we can express a complex number in polar form:

$$z = r(\cos\theta + j\sin\theta)$$

Using Equations (9.3) and (9.4) we can write

$$\begin{aligned} z &= r\left\{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)\right\} \\ &= r\left(1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} \cdots\right) \end{aligned}$$

Furthermore, we note that $e^{j\theta}$ can be written as

$$\begin{aligned} e^{j\theta} &= 1 + j\theta + \frac{j^2\theta^2}{2!} + \frac{j^3\theta^3}{3!} + \cdots \\ &= 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \cdots \end{aligned}$$

so that

$$z = r(\cos\theta + j\sin\theta) = re^{j\theta}$$

This is yet another form of the same complex number which we call the **exponential form**. We see that

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{9.5}$$

It is straightforward to show that

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

Therefore, if $\bar{z} = r(\cos\theta - \mathrm{j}\sin\theta)$ we can equivalently write $\bar{z} = r\mathrm{e}^{-\mathrm{j}\theta}$. The two expressions for $\mathrm{e}^{\mathrm{j}\theta}$ and $\mathrm{e}^{-\mathrm{j}\theta}$ are known as Euler's relations. From these it is easy to obtain the following useful results:

$$\cos\theta = \frac{\mathrm{e}^{\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}\theta}}{2} \qquad \sin\theta = \frac{\mathrm{e}^{\mathrm{j}\theta} - \mathrm{e}^{-\mathrm{j}\theta}}{2\mathrm{j}}$$

Example 9.16 We saw in Section 3.7 that a waveform can be written in the form $f(t) = A\cos(\omega t + \phi)$. Consider the complex number $\mathrm{e}^{\mathrm{j}(\omega t+\phi)}$. We can use Euler's relations to write

$$\mathrm{e}^{\mathrm{j}(\omega t+\phi)} = \cos(\omega t + \phi) + \mathrm{j}\sin(\omega t + \phi)$$

and hence,

$$f(t) = A\,\mathrm{Re}(\mathrm{e}^{\mathrm{j}(\omega t+\phi)})$$

EXERCISES 9.7

1 Find the modulus and argument of
(a) $3\mathrm{e}^{\mathrm{j}\pi/4}$ (b) $2\mathrm{e}^{-\mathrm{j}\pi/6}$ (c) $7\mathrm{e}^{\mathrm{j}\pi/3}$

2 Find the real and imaginary parts of
(a) $5\mathrm{e}^{\mathrm{j}\pi/3}$ (b) $\mathrm{e}^{\mathrm{j}2\pi/3}$
(c) $11\mathrm{e}^{\mathrm{j}\pi}$ (d) $2\mathrm{e}^{-\mathrm{j}\pi}$

3 Express $z = 6(\cos 30° + \mathrm{j}\sin 30°)$ in exponential form. Plot z on an Argand diagram and find its real and imaginary parts.

4 If $\sigma, \omega, T \in \mathbb{R}$, find the real and imaginary parts of $\mathrm{e}^{(\sigma+\mathrm{j}\omega)T}$.

5 Express $z = \mathrm{e}^{1+\mathrm{j}\pi/2}$ in the form $a + b\mathrm{j}$.

6 Express $-1 - \mathrm{j}$ in the form $r\mathrm{e}^{\mathrm{j}\theta}$.

7 Express
(a) $7 + 5\mathrm{j}$ and
(b) $\frac{1}{2} - \frac{1}{3}\mathrm{j}$ in exponential form.

8 Express $z_1 = 1 - \mathrm{j}$ and
$z_2 = \dfrac{1+\mathrm{j}}{\sqrt{3}-\mathrm{j}}$ in the form $r\mathrm{e}^{\mathrm{j}\theta}$.

Solutions

1 (a) $3, \pi/4$ (b) $2, -\pi/6$ (c) $7, \pi/3$

2 (a) 2.5, 4.3301 (b) −0.5, 0.8660 (c) −11, 0
(d) −2, 0

3 $6\mathrm{e}^{(\pi/6)\mathrm{j}}$, $\mathrm{Re}(z) = 5.1962$, $\mathrm{Im}(z) = 3$

4 Real part: $\mathrm{e}^{\sigma T}\cos\omega T$, imaginary part: $\mathrm{e}^{\sigma T}\sin\omega T$

5 $\mathrm{e}\mathrm{j}$

6 $\sqrt{2}\mathrm{e}^{(-3\pi/4)\mathrm{j}}$

7 (a) $\sqrt{74}\mathrm{e}^{0.62\mathrm{j}}$ (b) $0.60\,\mathrm{e}^{-0.59\mathrm{j}}$

8 $\sqrt{2}\mathrm{e}^{(-\pi/4)\mathrm{j}}$, $\dfrac{1}{\sqrt{2}}\mathrm{e}^{(5\pi/12)\mathrm{j}}$

9.8 PHASORS

Electrical engineers are often interested in analysing circuits in which there is an a.c. power supply. Almost invariably the supply waveform is sinusoidal and the resulting currents and voltages within the circuit are also sinusoidal. For example, a typical voltage is of the form

$$v(t) = V\cos(\omega t + \phi) = V\cos(2\pi f t + \phi) \tag{9.6}$$

where V is the maximum or peak value, ω is the angular frequency, f is the frequency, and ϕ is the phase relative to some reference waveform. This is known as the **time domain representation**. Each of the voltages and currents in the circuit has the same frequency as the supply but differs in magnitude and phase.

In order to analyse such circuits it is necessary to add, subtract, multiply and divide these waveforms. If the time domain representation is used then the mathematics becomes extremely tedious. An alternative approach is to introduce a waveform representation known as a **phasor**. A phasor is an entity consisting of two distinct parts: a magnitude and an angle. It is possible to represent a phasor by a complex number in polar form. The fixed magnitude of this complex number corresponds to the magnitude of the phasor and hence the amplitude of the waveform. The argument of this complex number, ϕ, corresponds to the angle of the phasor and hence the phase angle of the waveform. Figure 9.10 shows a phasor for the sinusoidal waveform of Equation (9.6).

The time dependency of the waveform is catered for by rotating the phasor anticlockwise at an angular frequency, ω. The projection of the phasor onto the real axis gives the instantaneous value of the waveform. However, the main interest of an engineer is in the phase relationships between the various sinusoids. Therefore the phasors are 'frozen' at a certain point in time. This may be chosen so that $t = 0$ or it may be chosen so that a convenient phasor, known as the **reference phasor**, aligns with the positive real axis. This approach is valid because the phase and magnitude relationships between the various phasors remain the same at all points in time once a circuit has recovered from any initial transients caused by switching.

Some textbooks refer to phasors as vectors. This can lead to confusion as it is possible to divide phasors whereas division of vectors by other vectors is not defined. In practice this is not a problem as phasors, although thought of as vectors, are manipulated as complex numbers, which can be divided. We will avoid these conceptual difficulties by introducing a different notation. We will denote a phasor by $\tilde{V}$, which corresponds to $V\angle\phi$ in complex number notation (see Figure 9.10). Thus, for example, a current $i(t) = I\cos(\omega t+\phi)$ would be written $\tilde{I}$ in phasor notation and $I\angle\phi$ in complex number notation.

Many engineers use the **root mean square** (r.m.s.) value of a sinusoid as the magnitude of a phasor. The justification for this is that it represents the value of a d.c. signal that would dissipate the same amount of power in a resistor as the sinusoid. For

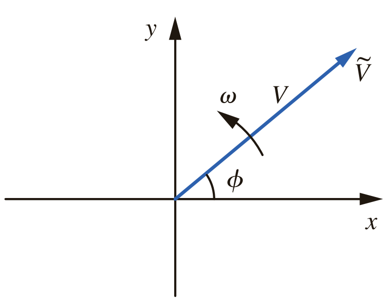

Figure 9.10
Illustration of the phasor $\tilde{V} = V\angle\phi$ where $\omega =$ angular frequency with which the phasor rotates.

example, in the case of a current signal, $I_{\mathrm{rms}}^2 R$ is the average power dissipated by the sinusoid $I\cos(\omega t + \phi)$ in a resistor R. For the case of a sinusoidal signal the r.m.s. value of the signal is $1/\sqrt{2}$ times the peak value of the signal (see Section 15.3, Example 15.4, for a proof of this). We will not adopt this approach but it is a common one.

We start by examining the phasor representation of individual circuit elements. In order to do this we need a phasor form of Ohm's law. This is

$$\tilde{V} = \tilde{I}Z \tag{9.7}$$

where $\tilde{V}$ is the voltage phasor, $\tilde{I}$ is the current phasor and Z is the impedance of an element or group of elements and may be a complex quantity. Note that phasors and complex numbers are mixed together in the same equation. This is a common practice because phasors are usually manipulated as complex numbers.

9.8.1 Resistor

Experimentally it can be shown that if an a.c. voltage is applied to a resistor then the current is in phase with the voltage. The ratio of the magnitude of the two waveforms is equal to the resistance, R. So, given $\tilde{I} = I\angle 0$, $Z = R\angle 0$, using Equation (9.7) we have $\tilde{V} = IR\angle 0$. This is illustrated in Figure 9.11.

9.8.2 Inductor

For an inductor we know from experiment that the voltage leads the current by a phase of $\pi/2$, and so the phase angle of the impedance is $\pi/2$. We also know that the magnitude of the impedance is given by ωL. So, given $\tilde{I} = I\angle 0$, $Z = \omega L\angle \pi/2$, using Equation (9.7) we have $\tilde{V} = I\omega L\angle \pi/2$. An alternative way of representing Z for an inductor is to use the Cartesian form, that is

$$\begin{aligned} Z = \omega L \mathrm{e}^{\mathrm{j}\pi/2} &= \omega L\left(\cos\frac{\pi}{2} + \mathrm{j}\sin\frac{\pi}{2}\right) \\ &= \mathrm{j}\omega L \end{aligned}$$

This is useful when phasors need to be added and subtracted. The phasor diagram for an inductor is illustrated in Figure 9.12.

9.8.3 Capacitor

For a capacitor it is known that the voltage lags the current by a phase of $\pi/2$ and the magnitude of the impedance is given by $\dfrac{1}{\omega C}$. So given $\tilde{I} = I\angle 0$, $Z = \dfrac{1}{\omega C}\angle -\pi/2$, we

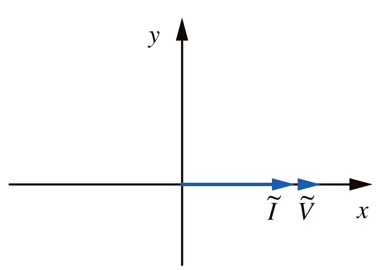

Figure 9.11
Phasor diagram for a resistor.

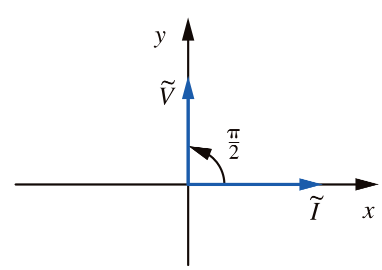

Figure 9.12
Phasor diagram for an inductor.

have, using Equation (9.7), $\tilde{V} = \dfrac{I}{\omega C}\angle -\pi/2$. Alternatively,

$$Z = \frac{e^{-j\pi/2}}{\omega C} = \frac{1}{\omega C}\left(\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right)$$
$$= -\frac{j}{\omega C}$$

Engineers often prefer to rewrite this last expression as

$$\frac{1}{j\omega C}$$

The phasor diagram for the capacitor is illustrated in Figure 9.13.

We have shown how phasors can be multiplied by a complex number; division is very similar. Addition of phasors will now be illustrated; subtraction is similar. Consider the circuit shown in Figure 9.14 in which a resistor, capacitor and inductor are connected in series and fed by an a.c. source. As this is a series circuit, the current through each element, $\tilde{I}$, is the same by Kirchhoff's current law. By Kirchhoff's voltage law the voltage rise produced by the supply, $\tilde{V}_S$, must equal the sum of the voltage drops across the elements. Therefore,

$$\tilde{V}_S = \tilde{V}_R + \tilde{V}_C + \tilde{V}_L$$

Note that this is an addition of phasors so that the voltage drops across the elements do not necessarily have the same phase. The phasor diagram for the circuit is shown in Figure 9.15, in this case with $|\tilde{V}_L| > |\tilde{V}_C|$; $\tilde{I}$ is the reference phasor.

Note that the phasor addition of the element voltages gives the overall supply voltage for a particular supply current. If the magnitude of these element voltage phasors is known then it is possible to calculate the supply voltage graphically. In practice it is easier to convert the polar form of the phasors into Cartesian form and use algebra to analyse the circuit.

Now,

$$\tilde{V}_R = \tilde{I}R\angle 0 = \tilde{I}R$$
$$\tilde{V}_L = \tilde{I}\omega L\angle \pi/2 = \tilde{I}j\omega L$$
$$\tilde{V}_C = \frac{\tilde{I}}{\omega C}\angle -\pi/2 = \frac{\tilde{I}}{j\omega C}$$

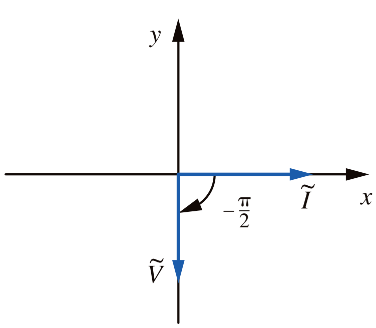

Figure 9.13
Phasor diagram for a capacitor.

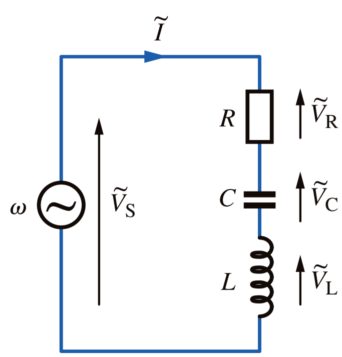

Figure 9.14
RLC circuit.

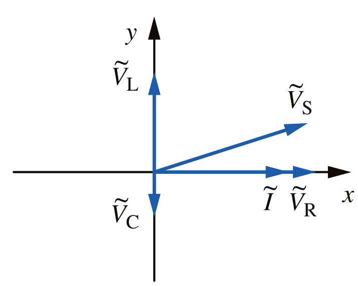

Figure 9.15
Phasor diagram for the circuit in Figure 9.14.

Therefore,

$$\begin{aligned}\tilde{V}_{\mathrm{S}} &= \tilde{V}_{\mathrm{R}} + \tilde{V}_{\mathrm{C}} + \tilde{V}_{\mathrm{L}} \\ &= \tilde{I}R + \tilde{I}\mathrm{j}\omega L + \frac{\tilde{I}}{\mathrm{j}\omega C} \\ &= \tilde{I}\left(R + \mathrm{j}\omega L + \frac{1}{\mathrm{j}\omega C}\right)\end{aligned}$$

Therefore the impedance of the circuit is $Z = R + \mathrm{j}\omega L + \dfrac{1}{\mathrm{j}\omega C}$. We can calculate the frequency for which the impedance of the circuit has minimum magnitude:

$$\begin{aligned}Z &= R + \mathrm{j}\omega L + \frac{1}{\mathrm{j}\omega C} \\ &= R + \mathrm{j}\omega L - \frac{\mathrm{j}}{\omega C} \\ &= R + \mathrm{j}\left(\omega L - \frac{1}{\omega C}\right)\end{aligned}$$

Now

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

and so, as ω varies $|Z|$ is a minimum when

$$\begin{aligned}\omega L - \frac{1}{\omega C} &= 0 \\ \omega^2 &= \frac{1}{LC} \\ \omega &= \sqrt{\frac{1}{LC}}\end{aligned}$$

This minimum value is $|Z| = R$. Examining Figure 9.15 it is clear that the minimum impedance occurs when $\tilde{V}_{\mathrm{L}}$ and $\tilde{V}_{\mathrm{C}}$ have the same magnitude, in which case $\tilde{V}_{\mathrm{S}}$ has no imaginary component. The frequency at which this occurs is known as the **resonant frequency** of the circuit.

Engineering application 9.1

The Poynting vector

An electromagnetic wave freely travelling in space has electric and magnetic field components which oscillate at right angles to each other and to the direction of propagation. This type of wave is known as a **transverse wave**. Figure 9.16 illustrates an electromagnetic wave travelling in free space. The **Poynting vector** is used to describe the energy flux associated with an electromagnetic wave. It has units of W m^{-2} and is a power per unit area, that is a power density. If the total power associated with an electromagnetic wave front is required then the Poynting vector can be integrated over an area of interest.

The Poynting vector, **S**, for electric field, **E**, and magnetic field, **H**, is defined as

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

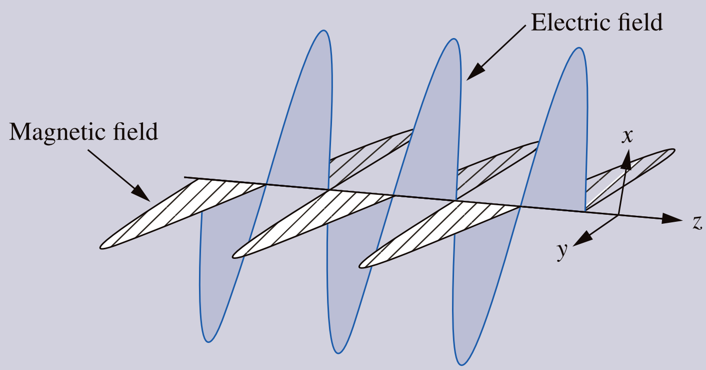

Figure 9.16
An electromagnetic wave travelling in free space.

where **E** and **H** are vector quantities. Note that **S** is also a vector quantity as this expression is a vector product.

It is possible to specify magnitude and phase of the field quantities using complex numbers; that is, in terms of **phasors**. Consider the case when **E** and **H** are the following:

$$\mathbf{E} = (2 + \mathrm{j}0.5)\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

$$\mathbf{H} = 0\mathbf{i} + (3.5 + \mathrm{j}0.25)\mathbf{j} + 0\mathbf{k}$$

Here Cartesian coordinates have been used to define the two field quantities and **i** is a unit vector in the x direction, **j** is a unit vector in the y direction and **k** is a unit vector in the z direction. Examining these two terms we note that the electric field is aligned in the x direction and the magnetic field is aligned in the y direction.

Calculating the Poynting vector we obtain

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 + \mathrm{j}0.5 & 0 & 0 \\ 0 & 3.5 + \mathrm{j}0.25 & 0 \end{vmatrix}$$

$$= 0\mathbf{i} - 0\mathbf{j} + (2 + \mathrm{j}0.5)(3.5 + \mathrm{j}0.25)\mathbf{k} = (6.875 + \mathrm{j}2.25)\mathbf{k}$$

We see that the only non-zero component is aligned in the z direction. This is consistent with our understanding of a transverse wave in that the direction of propagation is at right angles to the two field components. This implies that the energy flow of the electromagnetic wave is in the z direction.

9.9 DE MOIVRE'S THEOREM

A very important result in complex number theory is De Moivre's theorem which states that if $n \in \mathbb{N}$,

$$(\cos\theta + \mathrm{j}\sin\theta)^n = \cos n\theta + \mathrm{j}\sin n\theta \qquad (9.8)$$

Example 9.17 Verify De Moivre's theorem when $n = 1$ and $n = 2$.

Solution When $n = 1$, the theorem states:

$$(\cos\theta + \mathrm{j}\sin\theta)^1 = \cos 1\theta + \mathrm{j}\sin 1\theta$$

which is clearly true, and the theorem holds. When $n = 2$, we find

$$\begin{aligned}(\cos\theta + \mathrm{j}\sin\theta)^2 &= (\cos\theta + \mathrm{j}\sin\theta)(\cos\theta + \mathrm{j}\sin\theta)\\ &= \cos^2\theta + \mathrm{j}\sin\theta\cos\theta + \mathrm{j}\cos\theta\sin\theta + \mathrm{j}^2\sin^2\theta\\ &= \cos^2\theta - \sin^2\theta + \mathrm{j}(2\sin\theta\cos\theta)\end{aligned}$$

Recalling the trigonometric identities $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$, we can write the previous expression as

$$\cos 2\theta + \mathrm{j}\sin 2\theta$$

Therefore,

$$(\cos\theta + \mathrm{j}\sin\theta)^2 = \cos 2\theta + \mathrm{j}\sin 2\theta$$

and De Moivre's theorem has been verified when $n = 2$.

The theorem also holds when n is a rational number, that is $n = p/q$ where p and q are integers. Thus we have

$$\boxed{(\cos\theta + \mathrm{j}\sin\theta)^{p/q} = \cos\frac{p}{q}\theta + \mathrm{j}\sin\frac{p}{q}\theta}$$

In this form it can be used to obtain roots of complex numbers. For example,

$$\sqrt[3]{\cos\theta + \mathrm{j}\sin\theta} = (\cos\theta + \mathrm{j}\sin\theta)^{1/3} = \cos\frac{1}{3}\theta + \mathrm{j}\sin\frac{1}{3}\theta$$

In such a case the expression obtained is only one of the possible roots. Additional roots can be found as illustrated in Example 9.18.

De Moivre's theorem is particularly important for the solution of certain types of equation.

Example 9.18 Find all complex numbers z which satisfy

$$z^3 = 1 \tag{9.9}$$

Solution The solution of this equation is equivalent to finding the solutions of $z = 1^{1/3}$; that is, finding the cube roots of 1. Since we are allowing z to be complex, that is $z \in \mathbb{C}$, we can write

$$z = r(\cos\theta + \mathrm{j}\sin\theta)$$

Then, using De Moivre's theorem,

$$\begin{aligned}z^3 &= r^3(\cos\theta + \mathrm{j}\sin\theta)^3\\ &= r^3(\cos 3\theta + \mathrm{j}\sin 3\theta)\end{aligned}$$

We next convert the expression on the r.h.s. of Equation (9.9) into polar form. Figure 9.17 shows the number $1 = 1 + 0\mathrm{j}$ on an Argand diagram.

From the Argand diagram we see that its modulus is 1 and its argument is 0, or possibly $\pm 2\pi$, $\pm 4\pi$, ..., that is $2n\pi$ where $n \in \mathbb{Z}$. Consequently, we can write

$$1 = 1(\cos 2n\pi + \mathrm{j}\sin 2n\pi) \qquad n \in \mathbb{Z}$$

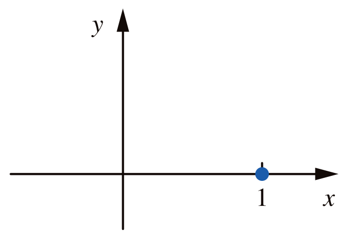

Figure 9.17
The complex number $z = 1 + 0\mathrm{j}$.

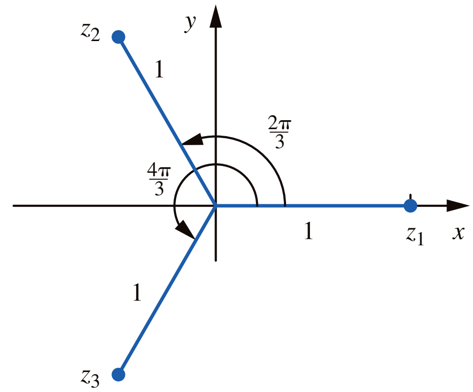

Figure 9.18
Solutions of $z^3 = 1$.

Using the polar form, Equation (9.9) becomes

$$r^3(\cos 3\theta + \mathrm{j}\sin 3\theta) = 1(\cos 2n\pi + \mathrm{j}\sin 2n\pi)$$

Comparing both sides of this equation we see that

$$r^3 = 1 \qquad \text{that is} \qquad r = 1 \text{ since } r \in \mathbb{R}$$

and

$$3\theta = 2n\pi \qquad \text{that is} \qquad \theta = 2n\pi/3 \qquad n \in \mathbb{Z}$$

Apparently θ can take infinitely many values, but, as we shall see, the corresponding complex numbers are simply repetitions. When $n = 0$, we find $\theta = 0$, so that

$$z = z_1 = 1(\cos 0 + \mathrm{j}\sin 0) = 1$$

is the first solution. When $n = 1$ we find $\theta = 2\pi/3$, so that

$$z = z_2 = 1\left(\cos\frac{2\pi}{3} + \mathrm{j}\sin\frac{2\pi}{3}\right) = -\frac{1}{2} + \mathrm{j}\frac{\sqrt{3}}{2}$$

is the second solution. When $n = 2$ we find $\theta = 4\pi/3$, so that

$$z = z_3 = 1\left(\cos\frac{4\pi}{3} + \mathrm{j}\sin\frac{4\pi}{3}\right) = -\frac{1}{2} - \mathrm{j}\frac{\sqrt{3}}{2}$$

is the third solution. If we continue searching for solutions using larger values of n we find that we only repeat solutions already obtained. It is often useful to plot solutions on an Argand diagram and this is easily done directly from the polar form, as shown in Figure 9.18. We note that the solutions are equally spaced at angles of $2\pi/3$.

Example 9.19 Find the complex numbers z which satisfy $z^2 = 4\mathrm{j}$.

Solution Since $z \in \mathbb{C}$ we write $z = r(\cos\theta + \mathrm{j}\sin\theta)$. Therefore,

$$\begin{aligned} z^2 &= r^2(\cos\theta + \mathrm{j}\sin\theta)^2 \\ &= r^2(\cos 2\theta + \mathrm{j}\sin 2\theta) \end{aligned}$$

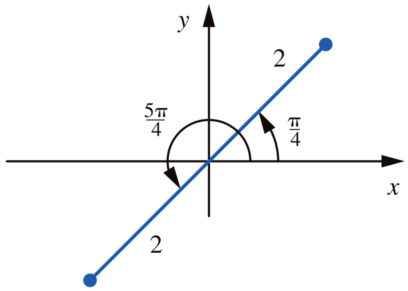

Figure 9.19
Solution of $z^2 = 4\text{j}$.

by De Moivre's theorem. Furthermore, 4j has modulus 4 and argument $\pi/2 + 2n\pi$, $n \in \mathbb{Z}$, that is

$$4\text{j} = 4\{\cos(\pi/2 + 2n\pi) + \text{j}\sin(\pi/2 + 2n\pi)\} \qquad n \in \mathbb{Z}$$

Therefore,

$$r^2(\cos 2\theta + \text{j}\sin 2\theta) = 4\{\cos(\pi/2 + 2n\pi) + \text{j}\sin(\pi/2 + 2n\pi)\}$$

Comparing both sides of this equation, we see that

$$r^2 = 4 \qquad \text{and so} \qquad r = 2$$

and

$$2\theta = \pi/2 + 2n\pi \qquad \text{and so} \qquad \theta = \pi/4 + n\pi$$

When $n = 0$ we find $\theta = \pi/4$, and when $n = 1$ we find $\theta = 5\pi/4$. Using larger values of n simply repeats solutions already obtained. These solutions are shown in Figure 9.19. We note that in this example the solutions are equally spaced at intervals of $2\pi/2 = \pi$. In Cartesian form,

$$z_1 = \frac{2}{\sqrt{2}} + \text{j}\frac{2}{\sqrt{2}} = \sqrt{2}(1 + \text{j}) \qquad \text{and} \qquad z_2 = -\frac{2}{\sqrt{2}} - \text{j}\frac{2}{\sqrt{2}} = -\sqrt{2}(1 + \text{j})$$

In general, the n roots of $z^n = a + \text{j}b$ are equally spaced at angles $2\pi/n$.

Once the technique for solving equations like those in Examples 9.18 and 9.19 has been mastered, engineers find it simpler to work with the abbreviated form $r\angle\theta$. Example 9.19 reworked in this fashion becomes

$$\text{Let } z = r\angle\theta, \qquad \text{then} \qquad z^2 = r^2\angle 2\theta$$

Furthermore, $4\text{j} = 4\angle\pi/2 + 2n\pi$, and hence if $z^2 = 4\text{j}$, we have

$$r^2\angle 2\theta = 4\angle\left(\frac{\pi}{2} + 2n\pi\right)$$

from which

$$r^2 = 4 \qquad \text{and} \qquad 2\theta = \frac{\pi}{2} + 2n\pi$$

as before. Rework Example 9.18 for yourself using this approach.

Engineering application 9.2

Characteristic impedance

In a later application we shall explain the concept of the **characteristic impedance** of an electrical transmission line in more detail. For now, we can assume that it is an important electrical parameter represented by the equation

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

where R, L, G and C are transmission line parameters and ω is the angular frequency, all of which are real numbers. If R and G are zero, as they would be for an ideal transmission line, then Z_0 is wholly real because:

$$Z_0 = \sqrt{\frac{j\omega L}{j\omega C}} = \sqrt{\frac{L}{C}}$$

Alternatively, if G or R is included then the equation involves taking the square root of a complex number.

Given values of R, G, L, C and ω we can write

$$\frac{R + j\omega L}{G + j\omega C}$$

in Cartesian form (see Example 9.9) and hence in polar form (see Section 9.5). Application of De Moivre's Theorem will then allow us to calculate the required square roots to find Z_0 .

Another application of De Moivre's theorem is the derivation of trigonometric identities.

Example 9.20 Use De Moivre's theorem to show that

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

and

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

Solution We know that

$$(\cos\theta + j\sin\theta)^3 = \cos 3\theta + j\sin 3\theta$$

Expanding the l.h.s. we find

$$\cos^3\theta + 3j\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - j\sin^3\theta = \cos 3\theta + j\sin 3\theta$$

Equating the real parts gives

$$\cos^3\theta - 3\cos\theta\sin^2\theta = \cos 3\theta \tag{9.10}$$

and equating the imaginary parts gives

$$3\cos^2\theta\sin\theta - \sin^3\theta = \sin 3\theta \tag{9.11}$$

Now, writing $\sin^2\theta = 1 - \cos^2\theta$ in Equation (9.10) and $\cos^2\theta = 1 - \sin^2\theta$ in Equation (9.11), we find

$$\begin{aligned}
\cos 3\theta &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) \\
&= \cos^3\theta + 3\cos^3\theta - 3\cos\theta \\
&= 4\cos^3\theta - 3\cos\theta \\
\sin 3\theta &= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta \\
&= 3\sin\theta - 4\sin^3\theta
\end{aligned}$$

as required.

This technique allows trigonometric functions of multiples of angles to be expressed in terms of powers. Sometimes we want to carry out the reverse process and express a power in terms of multiple angles. Consider Example 9.21.

Example 9.21 If $z = \cos\theta + \mathrm{j}\sin\theta$ show that

$$z + \frac{1}{z} = 2\cos\theta \qquad z - \frac{1}{z} = 2\mathrm{j}\sin\theta$$

and find similar expressions for $z^n + \dfrac{1}{z^n}$ and $z^n - \dfrac{1}{z^n}$.

Solution Consider the complex number

$$z = \cos\theta + \mathrm{j}\sin\theta$$

Using De Moivre's theorem,

$$\frac{1}{z} = z^{-1} = (\cos\theta + \mathrm{j}\sin\theta)^{-1} = \cos(-\theta) + \mathrm{j}\sin(-\theta)$$

But $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$, so that if $z = \cos\theta + \mathrm{j}\sin\theta$

$$\frac{1}{z} = \cos\theta - \mathrm{j}\sin\theta$$

Consequently,

$$z + \frac{1}{z} = 2\cos\theta \qquad \text{and} \qquad z - \frac{1}{z} = 2\mathrm{j}\sin\theta$$

Moreover,

$$z^n = \cos n\theta + \mathrm{j}\sin n\theta \qquad \text{and} \qquad z^{-n} = \cos n\theta - \mathrm{j}\sin n\theta$$

so that

$$z^n + \frac{1}{z^n} = 2\cos n\theta \qquad \text{and} \qquad z^n - \frac{1}{z^n} = 2\mathrm{j}\sin n\theta$$

$$z^n + \frac{1}{z^n} = 2\cos n\theta \qquad \text{and} \qquad z^n - \frac{1}{z^n} = 2\mathrm{j}\sin n\theta$$

Example 9.22 Show that $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$.

Solution The formulae obtained in Example 9.21 allow us to obtain expressions for powers of $\cos\theta$ and $\sin\theta$. Since $2\cos\theta = z + \frac{1}{z}$, squaring both sides we have

$$\begin{aligned} 2^2\cos^2\theta &= \left(z + \frac{1}{z}\right)^2 = z^2 + 2 + \frac{1}{z^2} \\ &= \left(z^2 + \frac{1}{z^2}\right) + 2 \end{aligned}$$

But $z^2 + \frac{1}{z^2} = 2\cos 2\theta$, so

$$2^2\cos^2\theta = 2\cos 2\theta + 2$$

and therefore,

$$\begin{aligned} \cos^2\theta &= \frac{1}{2}\cos 2\theta + \frac{1}{2} \\ &= \frac{1}{2}(\cos 2\theta + 1) \end{aligned}$$

as required.

EXERCISES 9.9

1 Express $(\cos\theta + \mathrm{j}\sin\theta)^9$ and $(\cos\theta + \mathrm{j}\sin\theta)^{1/2}$ in the form $\cos n\theta + \mathrm{j}\sin n\theta$.

2 Use De Moivre's theorem to simplify

(a) $(\cos 3\theta + \mathrm{j}\sin 3\theta)(\cos 4\theta + \mathrm{j}\sin 4\theta)$

(b) $\dfrac{\cos 8\theta + \mathrm{j}\sin 8\theta}{\cos 2\theta - \mathrm{j}\sin 2\theta}$

3 Solve the equations

(a) $z^3 + 1 = 0$

(b) $z^4 = 1 + \mathrm{j}$

4 Find $\sqrt[3]{2 + 2\mathrm{j}}$ and display your solutions on an Argand diagram.

5 Prove the following trigonometric identities:

(a) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

(b) $32\sin^6\theta = 10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta$

6 Solve the equation $z^4 + 25 = 0$.

7 Find the fifth roots of j and depict your solutions on an Argand diagram.

8 Show that $\cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$.

9 Show that $\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$.

10 Express $\cos 5\theta$ in terms of powers of $\cos\theta$.

11 Express $\sin 5\theta$ in terms of powers of $\sin\theta$.

12 Given $\mathrm{e}^{\mathrm{j}\theta} = \cos\theta + \mathrm{j}\sin\theta$, prove De Moivre's theorem in the form

$$(\mathrm{e}^{\mathrm{j}\theta})^n = \cos n\theta + \mathrm{j}\sin n\theta$$

Solutions

1 $\cos 9\theta + \mathrm{j}\sin 9\theta$, $\cos(\theta/2) + \mathrm{j}\sin(\theta/2)$

2 (a) $\cos 7\theta + \mathrm{j}\sin 7\theta$
(b) $\cos 10\theta + \mathrm{j}\sin 10\theta$

3 (a) $1\angle\pi/3 + 2n\pi/3 \quad n = 0, 1, 2$

(b) $2^{1/8}\angle\pi/16 + n\pi/2 \quad n = 0, 1, 2, 3$

4 $8^{1/6}\angle\pi/12 + 2n\pi/3 \quad n = 0, 1, 2$

6 $\sqrt{5}\angle \pi/4 + n\pi/2 \qquad n = 0, 1, 2, 3$

7 $1\angle \pi/10 + 2n\pi/5 \qquad n = 0, 1, 2, 3, 4$

10 $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

11 $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

9.10 LOCI AND REGIONS OF THE COMPLEX PLANE

Regions of the complex plane can often be conveniently described by means of complex numbers. For example, the points that lie on a circle of radius 2 centred at the origin (Figure 9.20) represent complex numbers all of which have a modulus of 2. The arguments are any value of θ, $-\pi < \theta \leqslant \pi$. We can describe all the points on this circle by the simple expression

$$|z| = 2$$

that is, all complex numbers with modulus 2. We say that the locus (or path) of the point z is a circle, radius 2, centred at the origin. The interior of the circle is described by $|z| < 2$ while its exterior is described by $|z| > 2$.

Similarly all points lying in the first quadrant (shaded in Figure 9.21) have arguments between 0 and $\pi/2$. This quadrant is therefore described by the expression:

$$0 < \arg(z) < \pi/2$$

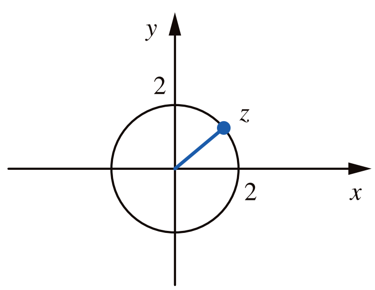

Figure 9.20
A circle, radius 2, centred at the origin.

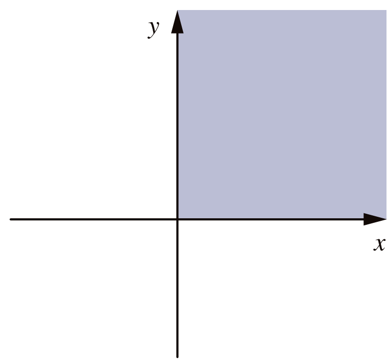

Figure 9.21
First quadrant of the x–y plane.

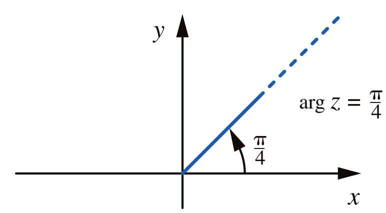

Figure 9.22
Locus of points satisfying $\arg(z) = \pi/4$.

Example 9.23 Sketch the locus of the point satisfying $\arg(z) = \pi/4$.

Solution The set of points with $\arg(z) = \pi/4$ comprises complex numbers whose argument is $\pi/4$. All these complex numbers lie on the line shown in Figure 9.22.

Example 9.24 Sketch the locus of the point satisfying $|z - 2| = 3$.

Solution First mark the fixed point 2 on the Argand diagram labelling it ‘A’ (Figure 9.23). Consider the complex number z represented by the point P. From the vector triangle law of addition

$$\vec{OA} + \vec{AP} = \vec{OP}$$
$$\vec{AP} = \vec{OP} - \vec{OA}$$

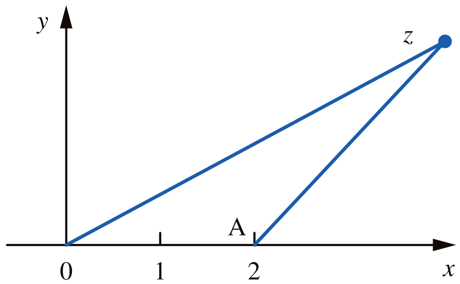

Figure 9.23
Points z and $2 + 0\text{j}$.

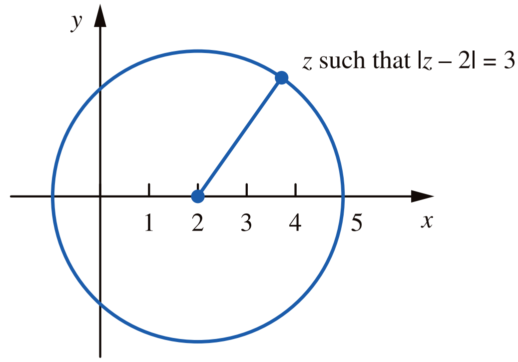

Figure 9.24
Locus of points satisfying $|z - 2| = 3$.

Recall from Section 9.6 that the graphical representation of the sum and difference of vectors in the plane, and the sum and difference of complex numbers, are equivalent. Since vector $\vec{OP}$ represents the complex number z, and vector $\vec{OA}$ represents the complex number 2, $\vec{AP} = \vec{OP} - \vec{OA}$ will represent $z - 2$. Therefore $|z - 2|$ represents the distance between A and P. We are given that $|z - 2| = 3$, which therefore means that P can be any point such that its distance from A is 3. This means that P can be any point on a circle of radius 3 centred at A(2, 0). The locus is shown in Figure 9.24. $|z - 2| < 3$ represents the interior of the circle while $|z - 2| > 3$ represents the exterior. Alternatively we can obtain the same result algebraically: given $|z - 2| = 3$ and also that $z = x + \text{j}y$, we can write

$$|z - 2| = |(x - 2) + \text{j}y| = 3$$

that is

$$\sqrt{(x - 2)^2 + y^2} = 3$$

or

$$(x - 2)^2 + y^2 = 9$$

Generally, the equation $(x - a)^2 + (y - b)^2 = r^2$ represents a circle of radius r centred at (a, b), so we see that $(x - 2)^2 + y^2 = 9$ represents a circle of radius 3 centred at (2, 0), as before.

Example 9.25 Use the algebraic approach to find the locus of the point z which satisfies

$$|z - 1| = \frac{1}{2}|z - \text{j}|$$

Solution If $z = x + \text{j}y$, then we have

$$|(x - 1) + \text{j}y| = \frac{1}{2}|x + \text{j}(y - 1)|$$

Therefore,

$$(x - 1)^2 + y^2 = \frac{1}{4}\{x^2 + (y - 1)^2\}$$

and so

$$4(x-1)^2 + 4y^2 = x^2 + (y-1)^2$$

that is

$$3x^2 - 8x + 3y^2 + 2y + 3 = 0$$

By completing the square this may be written in the form

$$3\left(x - \frac{4}{3}\right)^2 + 3\left(y + \frac{1}{3}\right)^2 = \frac{8}{3}$$

that is

$$\left(x - \frac{4}{3}\right)^2 + \left(y + \frac{1}{3}\right)^2 = \frac{8}{9}$$

which represents a circle of radius $\sqrt{8}/3$ centred at $\left(\frac{4}{3}, -\frac{1}{3}\right)$.

EXERCISES 9.10

1 Sketch the loci defined by

(a) $\arg(z) = 0$

(b) $\arg(z) = \pi/2$

(c) $\arg(z - 4) = \pi/4$

(d) $|2z| = |z - 1|$

2 Sketch the regions defined by

(a) $\mathrm{Re}(z) \geqslant 0$

(b) $\mathrm{Im}(z) < 3$

(c) $|z| > 3$

(d) $0 \leqslant \arg(z) \leqslant \pi/2$

(e) $|z + 2| \leqslant 3$

(f) $|z + \mathrm{j}| > 3$

(g) $|z - 1| < |z - 2|$

3 If $s = \sigma + \mathrm{j}\omega$ sketch the regions defined by

(a) $\sigma \leqslant 0$

(b) $\sigma \geqslant 0$

(c) $-2 \leqslant \omega \leqslant 2$

Solutions

1 See Figure S.19.

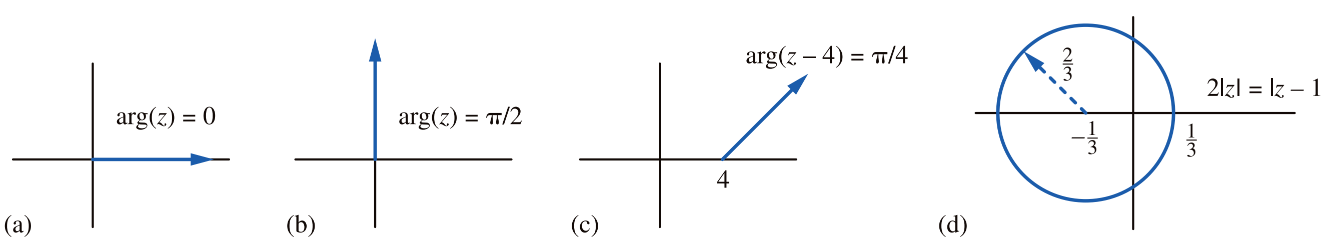

Figure S.19

2 See Figure S.20.

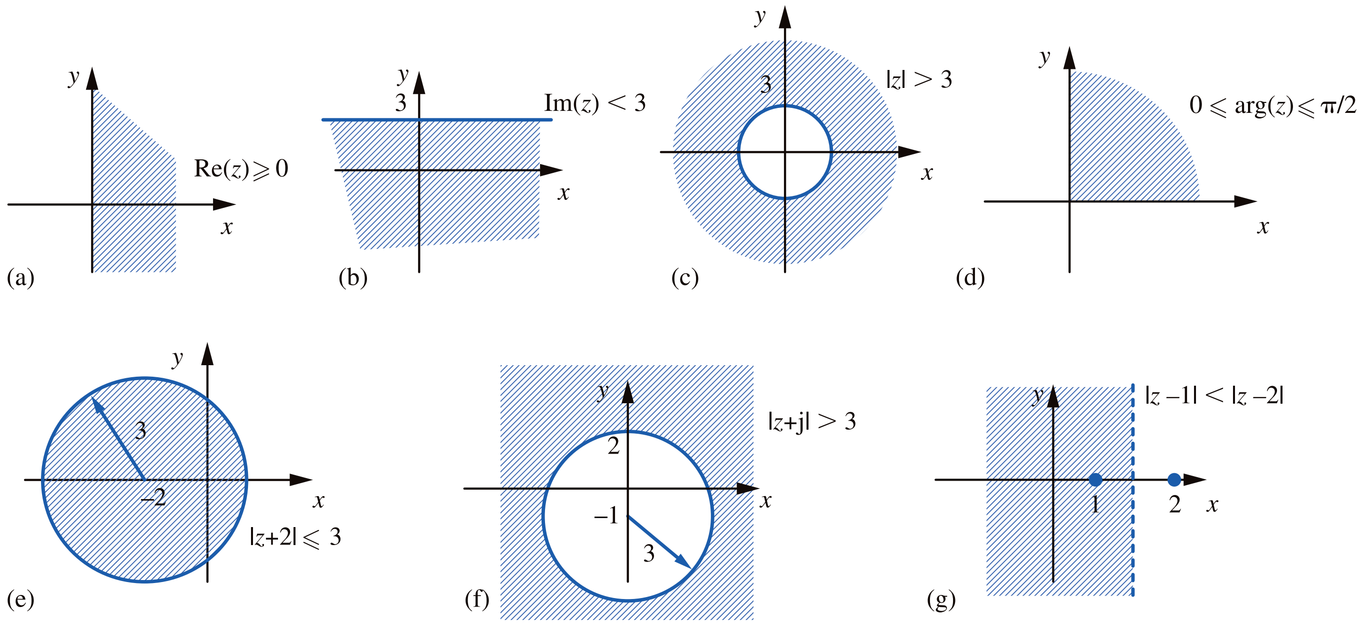

Figure S.20

3 See Figure S.21.

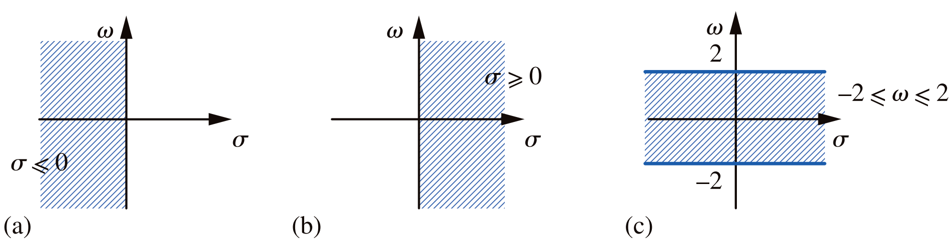

Figure S.21

REVIEW EXERCISES 9

1 Show that $\dfrac{1}{\cos\theta - \mathrm{j}\sin\theta} = \cos\theta + \mathrm{j}\sin\theta$.

2 Express in Cartesian form

(a) $\dfrac{5+4\mathrm{j}}{5-4\mathrm{j}}$ (b) $\dfrac{1}{2+3\mathrm{j}}$

(c) $\dfrac{1}{2+3\mathrm{j}} + \dfrac{1}{2-3\mathrm{j}}$ (d) $\dfrac{1}{x-\mathrm{j}y}$

3 Find the modulus and argument of $-\mathrm{j}$, -3, $1+\mathrm{j}$, $\cos\theta + \mathrm{j}\sin\theta$.

4 Mark on an Argand diagram vectors corresponding to the following complex numbers: $-3+2\mathrm{j}$, $-3-2\mathrm{j}$, $\cos\pi + \mathrm{j}\sin\pi$.

5 Express in the form $a + b\mathrm{j}$:

(a) $(\cos\theta + \mathrm{j}\sin\theta)^6$ (b) $\dfrac{1}{(\cos\theta + \mathrm{j}\sin\theta)^3}$

(c) $\dfrac{\cos\theta + \mathrm{j}\sin\theta}{\cos\phi + \mathrm{j}\sin\phi}$

6 If $z \in \mathbb{C}$, show that

(a) $z + \bar{z} = 2\mathrm{Re}(z)$ (b) $z - \bar{z} = 2\mathrm{j}\,\mathrm{Im}(z)$

(c) $z\bar{z} = |z|^2$

7 Show that $\mathrm{e}^{\mathrm{j}\omega t} + \mathrm{e}^{-\mathrm{j}\omega t} = 2\cos\omega t$ and find an expression for $\mathrm{e}^{\mathrm{j}\omega t} - \mathrm{e}^{-\mathrm{j}\omega t}$.

8 Express $1 + \mathrm{e}^{2\mathrm{j}\omega t}$ in the form $a + b\mathrm{j}$.

9 Sketch the region in the complex plane described by $|z + 2\mathrm{j}| < 1$.

10 Express $\mathrm{e}^{(1/2-6\mathrm{j})}$ in the form $a + b\mathrm{j}$.

11 Solve the equation $z^4 + 1 = \mathrm{j}\sqrt{3}$.

12 Express $s^2 + 6s + 13$ in the form $(s - a)(s - b)$ where $a, b \in \mathbb{C}$.

13 Express $2s^2 + 8s + 11$ in the form $2(s - a)(s - b)$ where $a, b \in \mathbb{C}$.

Solutions

2 (a) $\frac{9}{41} + \frac{40}{41}\mathrm{j}$ (b) $\frac{2}{13} - \frac{3}{13}\mathrm{j}$ (c) $\frac{4}{13}$

(d) $\frac{x}{x^2 + y^2} + \frac{y}{x^2 + y^2}\mathrm{j}$

3 $|-\mathrm{j}| = 1$, $\arg(-\mathrm{j}) = -\pi/2$; $|-3| = 3$, $\arg(-3) = \pi$; $|1 + \mathrm{j}| = \sqrt{2}$, $\arg(1 + \mathrm{j}) = \pi/4$, $|\cos\theta + \mathrm{j}\sin\theta| = 1$, $\arg(\cos\theta + \mathrm{j}\sin\theta) = \theta$

5 (a) $\cos 6\theta + \mathrm{j}\sin 6\theta$ (b) $\cos 3\theta - \mathrm{j}\sin 3\theta$

(c) $\cos(\theta - \phi) + \mathrm{j}\sin(\theta - \phi)$

7 $\mathrm{e}^{\mathrm{j}\omega t} - \mathrm{e}^{-\mathrm{j}\omega t} = 2\mathrm{j}\sin\omega t$

8 $1 + \cos 2\omega t + \mathrm{j}\sin 2\omega t$

10 $1.5831 + 0.4607\mathrm{j}$

11 $2^{1/4}\angle\pi/6 + n\pi/2 \qquad n = 0, 1, 2, 3$

12 $[s - (-3 + 2\mathrm{j})][s - (-3 - 2\mathrm{j})]$

13 $2\left[s - \left(-2 + \frac{\sqrt{6}}{2}\mathrm{j}\right)\right]\left[s - \left(-2 - \frac{\sqrt{6}}{2}\mathrm{j}\right)\right]$